

On a Weighted Singular Integral Operator with Shifts and Slowly Oscillating Data

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Abstract. Let α, β be orientation-preserving diffeomorphism (shifts) of $\mathbb{R}_+ = (0, \infty)$ onto itself with the only fixed points 0 and ∞ and U_α, U_β be the isometric shift operators on $L^p(\mathbb{R}_+)$ given by $U_\alpha f = (\alpha')^{1/p}(f \circ \alpha)$, $U_\beta f = (\beta')^{1/p}(f \circ \beta)$, and $P_2^\pm = (I \pm S_2)/2$ where

$$(S_2 f)(t) := \frac{1}{\pi i} \int_0^\infty \left(\frac{t}{\tau}\right)^{1/2-1/p} \frac{f(\tau)}{\tau-t} d\tau, \quad t \in \mathbb{R}_+,$$

is the weighted Cauchy singular integral operator. We prove that if α', β' and c, d are continuous on \mathbb{R}_+ and slowly oscillating at 0 and ∞ , and

$$\limsup_{t \rightarrow s} |c(t)| < 1, \quad \limsup_{t \rightarrow s} |d(t)| < 1, \quad s \in \{0, \infty\},$$

then the operator $(I - cU_\alpha)P_2^+ + (I - dU_\beta)P_2^-$ is Fredholm on $L^p(\mathbb{R}_+)$ and its index is equal to zero. Moreover, its regularizers are described.

Mathematics Subject Classification (2010). Primary 45E05;
Secondary 47A53, 47B35, 47G10, 47G30.

Keywords. Orientation-preserving shift, weighted Cauchy singular integral operator, slowly oscillating function, Fredholmness, index.

1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X , and let $\mathcal{K}(X)$ be the ideal of all compact operators in

This work was partially supported by the Fundação para a Ciéncia e a Tecnologia (Portuguese Foundation for Science and Technology) through the projects PEst-OE/MAT/UI0297/2014 (Centro de Matemática e Aplicações) and PEst-OE/MAT/UI4032/2014 (Centro de Análise Funcional e Aplicações). The second author was also supported by the CONACYT Project No. 168104 (México) and by PROMEP (México) via “Proyecto de Redes”.

$\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\text{Ind } A := \dim \ker A - \dim \ker A^*$$

is referred to as the *index* of A (see, e.g., [2, Sections 1.11–1.12], [4, Chap. 4]). For bounded linear operators A and B , we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$. Recall that an operator $B_r \in \mathcal{B}(X)$ (resp. $B_l \in \mathcal{B}(X)$) is said to be a right (resp. left) regularizer for A if

$$AB_r \simeq I \quad (\text{resp. } B_l A \simeq I).$$

It is well known that the operator A is Fredholm on X if and only if it admits simultaneously a right and a left regularizer. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [4, Chap. 4, Section 7]). Therefore we may speak of a regularizer $B = B_r = B_l$ of A and two different regularizers of A differ from each other by a compact operator.

A bounded continuous function f on $\mathbb{R}_+ = (0, \infty)$ is called slowly oscillating (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r \rightarrow s} \sup_{t, \tau \in [\lambda r, r]} |f(t) - f(\tau)| = 0, \quad s \in \{0, \infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^* -algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^* -algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$. Suppose α is an orientation-preserving diffeomorphism of \mathbb{R}_+ onto itself, which has only two fixed points 0 and ∞ . We say that α is a slowly oscillating shift if $\log \alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$.

Throughout the paper we suppose that $1 < p < \infty$. It is easily seen that if $\alpha \in SOS(\mathbb{R}_+)$, then the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ is bounded and invertible on all spaces $L^p(\mathbb{R}_+)$ and its inverse is given by $W_\alpha^{-1} = W_{\alpha^{-1}}$, where α^{-1} is the inverse function to α . Along with W_α we consider the weighted shift operator

$$U_\alpha := (\alpha')^{1/p} W_\alpha$$

being an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. Let S be the Cauchy singular integral operator given by

$$(Sf)(t) := \frac{1}{\pi i} \int_0^\infty \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R}_+,$$

where the integral is understood in the principal value sense. It is well known that S is bounded on $L^p(\mathbb{R}_+)$ for every $p \in (1, \infty)$. Let \mathcal{A} be the smallest closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ containing the identity operator I and the operator S . It is known (see, e.g., [3], [5, Section 2.1.2], [18, Sections 4.2.2–4.2.3], and [19]) that \mathcal{A} is commutative and for every $y \in (1, \infty)$ it contains

the weighted singular integral operator

$$(S_y f)(t) := \frac{1}{\pi i} \int_0^\infty \left(\frac{t}{\tau} \right)^{1/y-1/p} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R}_+,$$

and the operator with fixed singularities

$$(R_y f)(t) := \frac{1}{\pi i} \int_0^\infty \left(\frac{t}{\tau} \right)^{1/y-1/p} \frac{f(\tau)}{\tau + t} d\tau, \quad t \in \mathbb{R}_+,$$

which are understood in the principal value sense. For $y \in (1, \infty)$, put

$$P_y^\pm := (I \pm S_y)/2.$$

This paper is in some sense a continuation of our papers [7, 8, 9], where singular integral operators with shifts were studied under the mild assumptions that the coefficients belong to $SO(\mathbb{R}_+)$ and the shifts belong to $SOS(\mathbb{R}_+)$. In [7, 8] we found a Fredholm criterion for the singular integral operator

$$N = (aI - bW_\alpha)P_p^+ + (cI - dW_\alpha)P_p^-$$

with coefficients $a, b, c, d \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$. However, a formula for the calculation of the index of the operator N is still missing. Further, in [9] we proved that the operators

$$A_{ij} = U_\alpha^i P_p^+ + U_\beta^j P_p^-, \quad i, j \in \mathbb{Z},$$

with $\alpha, \beta \in SOS(\mathbb{R}_+)$ are all Fredholm and their indices are equal to zero. This result was the first step in the calculation of the index of N . Here we make the next step towards the calculation of the index of the operator N .

For $a \in SO(\mathbb{R})$, we will write $1 \gg a$ if

$$\limsup_{t \rightarrow s} |a(t)| < 1, \quad s \in \{0, \infty\}.$$

Theorem 1.1 (Main result). *Let $1 < p < \infty$ and $\alpha, \beta \in SOS(\mathbb{R}_+)$. Suppose $c, d \in SO(\mathbb{R}_+)$ are such that $1 \gg c$ and $1 \gg d$. Then the operator*

$$V := (I - cU_\alpha)P_2^+ + (I - dU_\beta)P_2^-,$$

is Fredholm on the space $L^p(\mathbb{R}_+)$ and $\text{Ind } V = 0$.

The paper is organized as follows. In Section 2 we collect necessary facts about slowly oscillating functions and slowly oscillating shifts, as well as about the invertibility of binomial functional operators $I - cU_\alpha$ with $c \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$ under the assumption that $1 \gg c$. Further we prove that the operators in the algebra \mathcal{A} commute modulo compact operators with the operators in the algebra $\mathcal{FO}_{\alpha, \beta}$ of functional operators with shifts and slowly oscillating data. Finally, we show that the ranges of two important continuous functions on \mathbb{R} do not contain the origin. In Section 3 we recall that the operators P_y^\pm and R_y , belonging to the algebra \mathcal{A} for every $y \in (1, \infty)$, can be viewed as Mellin convolution operators and formulate two relations between P_y^+ , P_y^- , and R_y . Section 4 contains results

on the boundedness, compactness of semi-commutators, and the Fredholmness of Mellin pseudodifferential operators with slowly oscillating symbols of limited smoothness (symbols in the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$). Results of this section are reformulations/modifications of corresponding results on Fourier pseudodifferential operators obtained by the second author in [12] (see also [13, 14, 15]). Notice that those results are further generalizations of earlier results by Rabinovich (see [17, Chap. 4] and the references therein) obtained for Mellin pseudodifferential operators with C^∞ slowly oscillating symbols.

In [9, Lemma 4.4] we proved that the operator $U_\gamma R_y$ with $\gamma \in SOS(\mathbb{R}_+)$ and $y \in (1, \infty)$ can be viewed as a Mellin pseudodifferential operator with a symbol in the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. In Section 5 we generalize that result and prove that $(I - vU_\gamma)R_y$ and $(I - vU_\gamma)^{-1}R_y$ with $y \in (1, \infty)$, $\gamma \in SOS(\mathbb{R}_+)$, and $v \in SO(\mathbb{R}_+)$ satisfying $1 \gg v$, can be viewed as Mellin pseudodifferential operators with symbols in the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. This is a key result in our analysis.

Section 6 is devoted to the proof of Theorem 1.1. Here we follow the idea, which was already used in a simpler situation of the operators A_{ij} in [9]. With the aid of results of Section 2 and Section 5, we will show that for every $\mu \in [0, 1]$ and $y \in (1, \infty)$, the operators

$$[(I - \mu cU_\alpha)P_y^+ + (I - \mu dU_\beta)P_y^-] \cdot [(I - \mu cU_\alpha)^{-1}P_y^+ + (I - \mu dU_\beta)^{-1}P_y^-],$$

$$[(I - \mu cU_\alpha)^{-1}P_y^+ + (I - \mu dU_\beta)^{-1}P_y^-] \cdot [(I - \mu cU_\alpha)P_y^+ + (I - \mu dU_\beta)P_y^-]$$

are equal up to compact summands to the same operator similar to a Mellin pseudodifferential operator with a symbol in the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Moreover, the latter pseudodifferential operator is Fredholm whenever $y = 2$ in view of results of Section 4. This will show that each operator

$$V_{\mu,2} = (I - \mu cU_\alpha)P_2^+ + (I - \mu dU_\beta)P_2^-$$

is Fredholm on $L^p(\mathbb{R}_+)$. Considering the homotopy $\mu \mapsto V_{\mu,2}$ for $\mu \in [0, 1]$, we see that the operator V is homotopic to the identity operator. Therefore, the index of V is equal to zero. This will complete the proof of Theorem 1.1.

As a by-product of the proof of the main result, in Section 7 we describe all regularizers of a slightly more general operator

$$W = (I - cU_\alpha^{\varepsilon_1})P_2^+ + (I - dU_\beta^{\varepsilon_2})P_2^-,$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and show that

$$G_y W \simeq R_y$$

for every $y \in (1, \infty)$, where G_y is an operator similar to a Mellin pseudodifferential operator with a symbol in $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ with some additional properties. The latter relation for $y = 2$ will play an important role in the proof of an index formula for the operator N in our forthcoming work [11].

2. Preliminaries

2.1. Fundamental Property of Slowly Oscillating Functions

For a unital commutative Banach algebra \mathfrak{A} , let $M(\mathfrak{A})$ denote its maximal ideal space. Identifying the points $t \in \overline{\mathbb{R}_+}$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}_+})$, we get $M(C(\overline{\mathbb{R}_+})) = \overline{\mathbb{R}_+}$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \{\xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}_+})} = s\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$. By [14, Proposition 2.1], the set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $(\text{clos}_{SO^*} \mathbb{R}_+) \setminus \mathbb{R}_+$ where $\text{clos}_{SO^*} \mathbb{R}_+$ is the weak-star closure of \mathbb{R}_+ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$. By [8, Lemma 2.2], the fibers $M_s(SO(\mathbb{R}_+))$ for $s \in \{0, \infty\}$ are connected compact Hausdorff spaces. In what follows we write

$$a(\xi) := \xi(a)$$

for every $a \in SO(\mathbb{R}_+)$ and every $\xi \in \Delta$.

Lemma 2.1 ([14, Proposition 2.2]). *Let $\{a_k\}_{k=1}^\infty$ be a countable subset of $SO(\mathbb{R}_+)$ and $s \in \{0, \infty\}$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \rightarrow s$ as $n \rightarrow \infty$ and*

$$a_k(\xi) = \lim_{n \rightarrow \infty} a_k(t_n) \quad \text{for all } k \in \mathbb{N}. \quad (2.1)$$

Conversely, if $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence such that $t_n \rightarrow s$ as $n \rightarrow \infty$, then there exists a functional $\xi \in M_s(SO(\mathbb{R}_+))$ such that (2.1) holds.

2.2. Slowly Oscillating Functions and Shifts

Repeating literally the proof of [6, Proposition 3.3], we obtain the following statement.

Lemma 2.2. *Suppose $\varphi \in C^1(\mathbb{R}_+)$ and put $\psi(t) := t\varphi'(t)$ for $t \in \mathbb{R}_+$. If $\varphi, \psi \in SO(\mathbb{R}_+)$, then*

$$\lim_{t \rightarrow s} \psi(t) = 0 \quad \text{for } s \in \{0, \infty\}.$$

Lemma 2.3 ([7, Lemma 2.2]). *An orientation-preserving shift $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to $SOS(\mathbb{R}_+)$ if and only if*

$$\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+,$$

for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that the function $t \mapsto t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and $\inf_{t \in \mathbb{R}_+} (1 + t\omega'(t)) > 0$.

Lemma 2.4 ([7, Lemma 2.3]). *If $c \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$, then $c \circ \alpha$ belongs to $SO(\mathbb{R}_+)$ and*

$$\lim_{t \rightarrow s} (c(t) - c[\alpha(t)]) = 0 \quad \text{for } s \in \{0, \infty\}.$$

For an orientation-preserving diffeomorphism $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, put

$$\alpha_0(t) := t, \quad \alpha_i(t) := \alpha[\alpha_{i-1}(t)], \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}_+.$$

Lemma 2.5 ([9, Corollary 2.5]). *If $\alpha, \beta \in SOS(\mathbb{R}_+)$, then $\alpha_i \circ \beta_j \in SOS(\mathbb{R}_+)$ for all $i, j \in \mathbb{Z}$.*

Lemma 2.6. *If $\alpha \in SOS(\mathbb{R}_+)$, then*

$$\omega(t) := \log[\alpha(t)/t], \quad \tilde{\omega}(t) := \log[\alpha_{-1}(t)/t], \quad t \in \mathbb{R}_+,$$

are slowly oscillating functions such that $\omega(\xi) = -\tilde{\omega}(\xi)$ for all $\xi \in \Delta$.

Proof. From Lemma 2.5 with $i = -1$ and $j = 0$ it follows that α_{-1} belongs to $SOS(\mathbb{R}_+)$. Then, by Lemma 2.3, $\omega, \tilde{\omega} \in SO(\mathbb{R}_+)$. It is easy to see that

$$\tilde{\omega}(t) = \log \frac{\alpha_{-1}(t)}{t} = -\log \frac{t}{\alpha_{-1}(t)} = -\log \frac{\alpha[\alpha_{-1}(t)]}{\alpha_{-1}(t)} = -\omega[\alpha_{-1}(t)]$$

for all $t \in \mathbb{R}_+$. Hence, from Lemma 2.4 it follows that $\omega \circ \alpha_{-1} \in SO(\mathbb{R}_+)$ and

$$\lim_{t \rightarrow s} (\omega(t) + \tilde{\omega}(t)) = \lim_{t \rightarrow s} (\omega(t) - \omega[\alpha_{-1}(t)]) = 0, \quad s \in \{0, \infty\}. \quad (2.2)$$

Fix $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. By Lemma 2.1, there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_j \rightarrow s$ and

$$\omega(\xi) = \lim_{j \rightarrow \infty} \omega(t_j), \quad \tilde{\omega}(\xi) = \lim_{j \rightarrow \infty} \tilde{\omega}(t_j). \quad (2.3)$$

From (2.2)–(2.3) we obtain

$$\omega(\xi) = \lim_{j \rightarrow \infty} \omega(t_j) - \lim_{j \rightarrow \infty} (\omega(t_j) + \tilde{\omega}(t_j)) = -\lim_{j \rightarrow \infty} \tilde{\omega}(t_j) = -\tilde{\omega}(\xi),$$

which completes the proof. \square

2.3. Invertibility of Binomial Functional Operators

From [7, Theorem 1.1] we immediately get the following.

Lemma 2.7. *Suppose $c \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. If $1 \gg c$, then the functional operator $I - cU_\alpha$ is invertible on the space $L^p(\mathbb{R}_+)$ and*

$$(I - cU_\alpha)^{-1} = \sum_{n=0}^{\infty} (cU_\alpha)^n.$$

2.4. Compactness of Commutators of SIO's and FO's

Let \mathfrak{B} be a Banach algebra and \mathfrak{S} be a subset of \mathfrak{B} . We denote by $\text{alg}_{\mathfrak{B}} \mathfrak{S}$ the smallest closed subalgebra of \mathfrak{B} containing \mathfrak{S} . Then

$$\mathcal{A} = \text{alg}_{\mathfrak{B}(L^p(\mathbb{R}_+))} \{I, S\}$$

is the algebra of singular integral operators (SIO's). Fix $\alpha, \beta \in SOS(\mathbb{R}_+)$ and consider the Banach algebra of functional operators (FO's) with shifts and slowly oscillating data defined by

$$\mathcal{FO}_{\alpha, \beta} := \text{alg}_{\mathfrak{B}(L^p(\mathbb{R}_+))} \{U_\alpha, U_\alpha^{-1}, U_\beta, U_\beta^{-1}, aI : a \in SO(\mathbb{R}_+)\}.$$

Lemma 2.8. *Let $\alpha, \beta \in SOS(\mathbb{R}_+)$. If $A \in \mathcal{FO}_{\alpha, \beta}$ and $B \in \mathcal{A}$, then*

$$AB - BA \in \mathcal{K}(L^p(\mathbb{R}_+)).$$

Proof. In view of [7, Corollary 6.4], we have $aB - BaI \in \mathcal{K}(L^p(\mathbb{R}_+))$ for all $a \in SO(\mathbb{R}_+)$ and all $B \in \mathcal{A}$. On the other hand, from [9, Lemma 2.7] it follows that $U_\gamma^{\pm 1}B - BU_\gamma^{\pm 1} \in \mathcal{K}(L^p(\mathbb{R}_+))$ for all $\gamma \in \{\alpha, \beta\}$ and $B \in \mathcal{A}$. Hence, $AB - BA \in \mathcal{K}(L^p(\mathbb{R}_+))$ for each generator A of $\mathcal{FO}_{\alpha, \beta}$ and each $B \in \mathcal{A}$. Thus, the same is true for all $A \in \mathcal{FO}_{\alpha, \beta}$ by a standard argument. \square

2.5. Ranges of Two Continuous Functions on \mathbb{R}

Given $a \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| \leq r\}$. For $x \in \mathbb{R}$, put

$$p_2^+(x) := \frac{e^{2\pi x}}{e^{2\pi x} + 1}, \quad p_2^-(x) := \frac{1}{e^{2\pi x} + 1}. \quad (2.4)$$

Lemma 2.9. *Let $\psi, \zeta \in \mathbb{R}$ and $v, w \in \mathbb{C}$. If*

$$f(x) := (1 - ve^{i\psi x})p_2^+(x) + (1 - we^{i\zeta x})p_2^-(x), \quad (2.5)$$

then $f(\mathbb{R}) \subset \mathbb{D}(1, r)$, where $r := \max(|v|, |w|)$.

Proof. From (2.4) and (2.5) we see that for every $x \in \mathbb{R}$ the point $f(x)$ lies on the line segment connecting the points $1 - ve^{i\psi x}$ and $1 - we^{i\zeta x}$. In turn, these points lie on the concentric circles

$$\{z \in \mathbb{C} : |z - 1| = |v|\}, \quad \{z \in \mathbb{C} : |z - 1| = |w|\}, \quad (2.6)$$

respectively. Thus, each line segment mentioned above is contained in the disk $\mathbb{D}(1, r) = \{z \in \mathbb{C} : |z - 1| \leq \max(|v|, |w|)\}$. \square

Lemma 2.10. *Let $\psi, \zeta \in \mathbb{R}$ and $v, w \in \mathbb{C}$ with $|v| < 1, |w| < 1$. If*

$$g(x) := (1 - ve^{i\psi x})^{-1}p_2^+(x) + (1 - we^{i\zeta x})^{-1}p_2^-(x), \quad x \in \mathbb{R}, \quad (2.7)$$

then $g(\mathbb{R}) \subset \mathbb{D}((1 - r^2)^{-1}, (1 - r^2)^{-1}r)$, where $r = \max(|v|, |w|) < 1$.

Proof. From (2.4) and (2.7) we see that for every $x \in \mathbb{R}$ the point $g(x)$ lies on the line segment connecting the points $(1 - ve^{i\psi x})^{-1}$ and $(1 - we^{i\zeta x})^{-1}$. In turn, these points lie on the images of the circles given by (2.6) under the inversion mapping $z \mapsto 1/z$. The image of the first circle in (2.6) is the circle $\mathbb{T}_v := \{z \in \mathbb{C} : |z - b| = \rho\}$ with center and radius given by

$$b = [(1 - |v|)^{-1} + (1 + |v|)^{-1}]/2 = (1 - |v|^2)^{-1},$$

$$\rho = [(1 - |v|)^{-1} - (1 + |v|)^{-1}]/2 = (1 - |v|^2)^{-1}|v|.$$

Analogously, the image of the second circle in (2.6) is the circle

$$\mathbb{T}_w := \{z \in \mathbb{C} : |z - (1 - |w|^2)^{-1}| = (1 - |w|^2)^{-1}|w|\}.$$

Let \mathbb{D}_v and \mathbb{D}_w be the closed disks whose boundaries are \mathbb{T}_v and \mathbb{T}_w , respectively. Obviously, one of these disks is contained in another one, namely, $\mathbb{D}_v \subset \mathbb{D}_w$ if $|v| \leq |w|$ and $\mathbb{D}_w \subset \mathbb{D}_v$ otherwise. Then each point $g(x)$, lying on the segment connecting the points $(1 - ve^{i\psi x})^{-1} \in \mathbb{T}_v$ and $(1 - we^{i\zeta x})^{-1} \in \mathbb{T}_w$, belongs to the biggest disk in $\{\mathbb{D}_v, \mathbb{D}_w\}$, that is, to the disk with center $(1 - r^2)^{-1}$ and radius $(1 - r^2)^{-1}r$, where $r = \max(|v|, |w|) < 1$. \square

From Lemmas 2.9 and 2.10 it follows that the ranges $f(\mathbb{R})$ and $g(\mathbb{R})$ do not contain the origin if $|v| < 1$ and $|w| < 1$.

3. Weighted Singular Integral Operators Are Similar to Mellin Convolution Operators

3.1. Mellin Convolution Operators

Let $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ denote the Fourier transform,

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(y) e^{-ixy} dy, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the inverse of \mathcal{F} . A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R})$ if the mapping $f \mapsto \mathcal{F}^{-1}a\mathcal{F}f$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ onto itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $\mathcal{M}_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $\mathcal{M}_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{\mathcal{M}_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on \mathbb{R}_+ . Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathcal{M} : L^2(\mathbb{R}_+, d\mu) \rightarrow L^2(\mathbb{R}), \quad (\mathcal{M}f)(x) := \int_{\mathbb{R}_+} f(t) t^{-ix} \frac{dt}{t}.$$

It is an invertible operator, with inverse given by

$$\mathcal{M}^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}_+, d\mu), \quad (\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x) t^{ix} dx.$$

Let E be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \rightarrow L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \quad (3.1)$$

Then the map $A \mapsto E^{-1}AE$ transforms the Fourier convolution operator $W^0(a) = \mathcal{F}^{-1}a\mathcal{F}$ to the Mellin convolution operator

$$\text{Co}(a) := \mathcal{M}^{-1}a\mathcal{M}$$

with the same symbol a . Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

3.2. Algebra \mathcal{A} of Singular Integral Operators

Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t), \quad t \in \mathbb{R}_+, \quad (3.2)$$

The following statement is well known (see, e.g., [3], [5, Section 2.1.2], and [18, Sections 4.2.2–4.2.3]).

Lemma 3.1. *For every $y \in (1, \infty)$, the functions s_y and r_y given by*

$$s_y(x) := \coth[\pi(x + i/y)], \quad r_y(x) := 1/\sinh[\pi(x + i/y)], \quad x \in \mathbb{R},$$

belong to $\mathcal{M}_p(\mathbb{R})$, the operators S_y and R_y belong to the algebra \mathcal{A} , and

$$S_y = \Phi^{-1} \operatorname{Co}(s_y)\Phi, \quad R_y = \Phi^{-1} \operatorname{Co}(r_y)\Phi.$$

For $y \in (1, \infty)$ and $x \in \mathbb{R}$, put

$$p_y^\pm(x) := (1 \pm s_y(x))/2.$$

This definition is consistent with (2.4) because $s_2(x) = \tanh(\pi x)$ for $x \in \mathbb{R}$. In view of Lemma 3.1 we have

$$P_y^\pm = (I \pm S_y)/2 = \Phi^{-1} \operatorname{Co}(p_y^\pm)\Phi.$$

Lemma 3.2. (a) For $y \in (1, \infty)$ and $x \in \mathbb{R}$, we have

$$p_y^+(x)p_y^-(x) = -\frac{(r_y(x))^2}{4}, \quad (p_y^\pm(x))^2 = p_y^\pm(x) + \frac{(r_y(x))^2}{4}.$$

(b) For every $y \in \mathbb{R}_+$, we have

$$P_y^+ P_y^- = P_y^- P_y^+ = -\frac{R_y^2}{4}, \quad (P_y^\pm)^2 = P_y^\pm + \frac{R_y^2}{4}.$$

Proof. Part (a) follows straightforwardly from the identity $s_y^2(x) - r_y^2(x) = 1$. Part (b) follows from part (a) and Lemma 3.1. \square

4. Mellin Pseudodifferential Operators and Their Symbols

4.1. Boundedness of Mellin Pseudodifferential Operators

In 1991 Rabinovich [16] proposed to use Mellin pseudodifferential operators with C^∞ slowly oscillating symbols to study singular integral operators with slowly oscillating coefficients on L^p spaces. This idea was exploited in a series of papers by Rabinovich and coauthors. A detailed history and a complete bibliography up to 2004 can be found in [17, Sections 4.6–4.7]. Further, the second author developed in [12] a handy for our purposes theory of Fourier pseudodifferential operators with slowly oscillating symbols of limited smoothness (much less restrictive than in the works mentioned in [17]). In this section we translate necessary results from [12] to the Mellin setting with the aid of the transformation

$$A \mapsto E^{-1}AE,$$

where $A \in \mathcal{B}(L^p(\mathbb{R}))$ and the isometric isomorphism $E : L^p(\mathbb{R}_+, d\mu) \rightarrow L^p(\mathbb{R})$ is defined by (3.1).

Let a be an absolutely continuous function of finite total variation

$$V(a) := \int_{\mathbb{R}} |a'(x)|dx$$

on \mathbb{R} . The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on \mathbb{R} becomes a Banach algebra equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a). \tag{4.1}$$

Following [12, 13], let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$ -valued functions on \mathbb{R}_+ with the norm

$$\|\mathbf{a}(\cdot, \cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} = \sup_{t \in \mathbb{R}_+} \|\mathbf{a}(t, \cdot)\|_V.$$

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on \mathbb{R}_+ .

The following boundedness result for Mellin pseudodifferential operators follows from [13, Theorem 6.1] (see also [12, Theorem 3.1]).

Theorem 4.1. *If $\mathbf{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\text{Op}(\mathbf{a})$, defined for functions $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral*

$$[\text{Op}(\mathbf{a})f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathbf{a}(t, x) \left(\frac{t}{\tau} \right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for } t \in \mathbb{R}_+,$$

extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ and there is a number $C_p \in (0, \infty)$ depending only on p such that

$$\|\text{Op}(\mathbf{a})\|_{\mathcal{B}(L^p(\mathbb{R}_+, d\mu))} \leq C_p \|\mathbf{a}\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}.$$

Obviously, if $\mathbf{a}(t, x) = a(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then the Mellin pseudodifferential operator $\text{Op}(\mathbf{a})$ becomes the Mellin convolution operator

$$\text{Op}(\mathbf{a}) = \text{Co}(a).$$

4.2. Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$

Let $SO(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$ -valued functions \mathbf{a} on \mathbb{R}_+ that slowly oscillate at 0 and ∞ , that is,

$$\lim_{r \rightarrow 0} \text{cm}_r^C(\mathbf{a}) = \lim_{r \rightarrow \infty} \text{cm}_r^C(\mathbf{a}) = 0,$$

where

$$\text{cm}_r^C(\mathbf{a}) := \max \left\{ \|\mathbf{a}(t, \cdot) - \mathbf{a}(\tau, \cdot)\|_{L^\infty(\mathbb{R})} : t, \tau \in [r, 2r] \right\}.$$

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$ -valued functions $\mathbf{a} \in SO(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$\lim_{|h| \rightarrow 0} \sup_{t \in \mathbb{R}_+} \|\mathbf{a}(t, \cdot) - \mathbf{a}^h(t, \cdot)\|_V = 0$$

where $\mathbf{a}^h(t, x) := \mathbf{a}(t, x + h)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Let $\mathbf{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $\mathbf{a}(t, \cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm\infty$, which will be denoted by $\mathbf{a}(t, \pm\infty)$. Now we explain how to extend the function \mathbf{a} to $\Delta \times \overline{\mathbb{R}}$. By analogy with [12, Lemma 2.7] with the aid of Lemma 2.1 one can prove the following.

Lemma 4.2. *Let $s \in \{0, \infty\}$ and $\{\mathbf{a}_k\}_{k=1}^\infty$ be a countable subset of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ and functions $\mathbf{a}_k(\xi, \cdot) \in V(\mathbb{R})$ such that $t_j \rightarrow s$ as $j \rightarrow \infty$ and*

$$\mathbf{a}_k(\xi, x) = \lim_{j \rightarrow \infty} \mathbf{a}_k(t_j, x)$$

for every $x \in \overline{\mathbb{R}}$ and every $k \in \mathbb{N}$.

A straightforward application of Lemma 4.2 leads to the following.

Lemma 4.3. *Let $\mathfrak{b} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, $m, n \in \mathbb{N}$, and $\mathfrak{a}_{ij} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. If*

$$\mathfrak{b}(t, x) = \sum_{i=1}^m \prod_{j=1}^n \mathfrak{a}_{ij}(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

then

$$\mathfrak{b}(\xi, x) = \sum_{i=1}^m \prod_{j=1}^n \mathfrak{a}_{ij}(\xi, x), \quad (\xi, x) \in \Delta \times \overline{\mathbb{R}}.$$

Lemma 4.4 ([10, Lemma 3.2]). *Let $\{\mathfrak{a}_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that the series $\sum_{n=1}^{\infty} \mathfrak{a}_n$ converges in the norm of the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ to a function $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Then*

$$\mathfrak{a}(t, \pm\infty) = \sum_{n=1}^{\infty} \mathfrak{a}_n(t, \pm\infty) \quad \text{for all } t \in \mathbb{R}_+, \quad (4.2)$$

$$\mathfrak{a}(\xi, x) = \sum_{n=1}^{\infty} \mathfrak{a}_n(\xi, x) \quad \text{for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (4.3)$$

4.3. Products of Mellin Pseudodifferential Operators

Applying the relation

$$\text{Op}(\mathfrak{a}) = E^{-1}a(x, D)E \quad (4.4)$$

between the Mellin pseudodifferential operator $\text{Op}(\mathfrak{a})$ and the Fourier pseudodifferential operator $a(x, D)$ considered in [12], where

$$\mathfrak{a}(t, x) = a(\ln t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (4.5)$$

and E is given by (3.1), we infer from [12, Theorem 8.3] the following compactness result.

Theorem 4.5. *If $\mathfrak{a}, \mathfrak{b} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, then*

$$\text{Op}(\mathfrak{a}) \text{Op}(\mathfrak{b}) \simeq \text{Op}(\mathfrak{a}\mathfrak{b}).$$

From (3.1), (4.4)–(4.5), [12, Lemmas 7.1, 7.2], and the proof of [12, Lemma 8.1] we can extract the following.

Lemma 4.6. *If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ are such that \mathfrak{a} depends only on the first variable and \mathfrak{c} depends only on the second variable, then*

$$\text{Op}(\mathfrak{a}) \text{Op}(\mathfrak{b}) \text{Op}(\mathfrak{c}) = \text{Op}(\mathfrak{a}\mathfrak{b}\mathfrak{c}).$$

4.4. Fredholmness of Mellin Pseudodifferential Operators

To study the Fredholmness of Mellin pseudodifferential operators, we need the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all functions \mathfrak{a} belonging to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{m \rightarrow \infty} \sup_{t \in \mathbb{R}_+} \int_{\mathbb{R} \setminus [-m, m]} \left| \frac{\partial \mathfrak{a}(t, x)}{\partial x} \right| dx = 0.$$

Now we are in a position to formulate the main result of this section.

Theorem 4.7 ([10, Theorem 5.8]). *Suppose $\mathfrak{a} \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.*

(a) *If the Mellin pseudodifferential operator $\text{Op}(\mathfrak{a})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$, then*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \text{ for all } t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi, x) \neq 0 \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \quad (4.6)$$

(b) *If (4.6) holds, then the Mellin pseudodifferential operator $\text{Op}(\mathfrak{a})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ and each its regularizer has the form $\text{Op}(\mathfrak{b}) + K$, where K is a compact operator on the space $L^p(\mathbb{R}_+, d\mu)$ and $\mathfrak{b} \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that*

$$\mathfrak{b}(t, \pm\infty) = 1/\mathfrak{a}(t, \pm\infty) \text{ for all } t \in \mathbb{R}_+,$$

$$\mathfrak{b}(\xi, x) = 1/\mathfrak{a}(\xi, x) \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}.$$

Note that part (a) follows from [15, Theorem 4.3] and part (b) is the main result of [10].

5. Applications of Mellin Pseudodifferential Operators

5.1. Some Important Functions in the Algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

Lemma 5.1 ([9, Lemma 4.2]). *Let $g \in SO(\mathbb{R}_+)$. Then for every $y \in (1, \infty)$ the functions*

$$\mathfrak{g}(t, x) := g(t), \quad \mathfrak{s}_y(t, x) := s_y(x), \quad \mathfrak{r}_y(t, x) := r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the Banach algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Lemma 5.2 ([9, Lemma 4.3]). *Suppose $\omega \in SO(\mathbb{R}_+)$ is a real-valued function. Then for every $y \in (1, \infty)$ the function*

$$\mathfrak{b}(t, x) := e^{i\omega(t)x} r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

belongs to the Banach algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and there is a positive constant $C(y)$ depending only on y such that

$$\|\mathfrak{b}\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \leq C(y) \left(1 + \sup_{t \in \mathbb{R}_+} |\omega(t)| \right).$$

5.2. Operator $U_\gamma R_y$

Lemma 5.3 ([9, Lemma 4.4]). *Let $\gamma \in SOS(\mathbb{R}_+)$ and U_γ be the associated isometric shift operator on $L^p(\mathbb{R}_+)$. For every $y \in (1, \infty)$, the operator $U_\gamma R_y$ can be realized as the Mellin pseudodifferential operator:*

$$U_\gamma R_y = \Phi^{-1} \text{Op}(\mathfrak{d}) \Phi,$$

where the function \mathfrak{d} , given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{d}(t, x) := (1 + t\psi'(t))^{1/p} e^{i\psi(t)x} r_y(x) \quad \text{with} \quad \psi(t) := \log[\gamma(t)/t],$$

belongs to the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

5.3. Operator $(I - vU_\gamma)R_y$

The previous lemma can be easily generalized to the case of operators containing slowly oscillating coefficients.

Lemma 5.4. *Let $y \in (1, \infty)$, $v \in SO(\mathbb{R}_+)$, and $\gamma \in SOS(\mathbb{R}_+)$. Then*

(a) *the operator $(I - vU_\gamma)R_y$ can be realized as the Mellin pseudodifferential operator:*

$$(I - vU_\gamma)R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{a})\Phi,$$

where the function \mathfrak{a} , given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{a}(t, x) := (1 - v(t))(\Psi(t))^{1/p} e^{i\psi(t)x} r_y(x)$$

with $\psi(t) := \log[\gamma(t)/t]$ and $\Psi(t) := 1 + t\psi'(t)$, belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$;

(b) *we have*

$$\mathfrak{a}(\xi, x) = \begin{cases} (1 - v(\xi))e^{i\psi(\xi)x} r_y(x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}. \end{cases}$$

Proof. (a) This statement follows straightforwardly from Lemmas 5.1, 5.3, and 4.6.

(b) If $t \in \mathbb{R}_+$, then obviously

$$\mathfrak{a}(t, x) = 0 \quad \text{for } x \in \{\pm\infty\}. \quad (5.1)$$

By Lemma 2.3, $\psi \in SO(\mathbb{R}_+)$. Since $v, \psi \in SO(\mathbb{R}_+)$, from Lemma 5.1 it follows that the functions

$$\mathfrak{v}(t, x) := v(t), \quad \tilde{\psi}(t, x) := \psi(t), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.2)$$

belong to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Consider the finite family $\{\mathfrak{a}, \mathfrak{v}, \tilde{\psi}\} \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Fix $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. By Lemma 4.2 and (5.2), there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ and a function $\mathfrak{a}(\xi, \cdot) \in V(\mathbb{R}_+)$ such that

$$\lim_{j \rightarrow \infty} t_j = s, \quad v(\xi) = \lim_{j \rightarrow \infty} v(t_j), \quad \psi(\xi) = \lim_{j \rightarrow \infty} \psi(t_j), \quad (5.3)$$

$$\mathfrak{a}(\xi, x) = \lim_{j \rightarrow \infty} \mathfrak{a}(t_j, x), \quad x \in \mathbb{R}. \quad (5.4)$$

From Lemmas 2.2 and 2.3 we obtain

$$\lim_{j \rightarrow \infty} (\Psi(t_j))^{1/p} = 1. \quad (5.5)$$

From (5.1) and (5.4) we get

$$\mathfrak{a}(\xi, x) = 0 \quad \text{for } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}.$$

Finally, from (5.3)–(5.5) we obtain for $(\xi, x) \in \Delta \times \mathbb{R}$,

$$\begin{aligned} \mathfrak{a}(\xi, x) &= \lim_{j \rightarrow \infty} \mathfrak{a}(t_j, x) \\ &= \left(1 - \left(\lim_{j \rightarrow \infty} v(t_j) \right) \left(\lim_{j \rightarrow \infty} (\Psi(t_j))^{1/p} \right) \exp \left(ix \lim_{j \rightarrow \infty} \psi(t_j) \right) \right) r_y(x) \\ &= (1 - v(\xi))e^{i\psi(\xi)x} r_y(x), \end{aligned}$$

which completes the proof. \square

5.4. Operator $(I - vU_\gamma)^{-1}R_y$

The following statement is crucial for our analysis. It says that the operators $(I - vU_\gamma)R_y$ and $(I - vU_\gamma)^{-1}R_y$ have similar nature.

Lemma 5.5. *Let $y \in (1, \infty)$, $v \in SO(\mathbb{R}_+)$, and $\gamma \in SOS(\mathbb{R}_+)$. If $1 \gg v$, then*

- (a) *the operator $A := I - vU_\gamma$ is invertible on $L^p(\mathbb{R}_+)$;*
- (b) *the operator $A^{-1}R_y$ can be realized as the Mellin pseudodifferential operator:*

$$A^{-1}R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{c})\Phi, \quad (5.6)$$

where the function \mathfrak{c} , given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}(t, x) := r_y(x) + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} v[\gamma_k(t)] (\Psi[\gamma_k(t)])^{1/p} e^{i\psi[\gamma_k(t)]x} \right) r_y(x) \quad (5.7)$$

with $\psi(t) := \log[\gamma(t)/t]$ and $\Psi(t) := 1 + t\psi'(t)$, belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$;

- (c) *we have*

$$\mathfrak{c}(\xi, x) = \begin{cases} (1 - v(\xi)e^{i\psi(\xi)x})^{-1}r_y(x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}. \end{cases}$$

Proof. (a) Since $1 \gg v$, from Lemma 2.7 we conclude that A is invertible on the space $L^p(\mathbb{R}_+)$ and

$$A^{-1} = \sum_{n=0}^{\infty} (vU_\gamma)^n. \quad (5.8)$$

Part (a) is proved.

- (b) By Lemmas 3.1 and 5.1,

$$R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{c}_0)\Phi, \quad (5.9)$$

where the function \mathfrak{c}_0 , given by

$$\mathfrak{c}_0(t, x) := r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.10)$$

belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

If $\gamma \in SOS(\mathbb{R}_+)$, then from Lemma 2.5 it follows that $\gamma_n \in SOS(\mathbb{R}_+)$ for every $n \in \mathbb{Z}$. By Lemma 2.3, the functions

$$\psi_n(t) := \log \frac{\gamma_n(t)}{t}, \quad \Psi_n(t) := 1 + t\psi'_n(t) \quad t \in \mathbb{R}_+, \quad n \in \mathbb{Z}, \quad (5.11)$$

are real-valued functions in $SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$. For every $n \in \mathbb{N}$,

$$(vU_\gamma)^n R_y = \left(\prod_{k=0}^{n-1} v \circ \gamma_k \right) U_{\gamma_n} R_y. \quad (5.12)$$

By Lemma 5.3,

$$U_{\gamma_n} R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{d}_n)\Phi, \quad (5.13)$$

where the function \mathfrak{d}_n , given by

$$\mathfrak{d}_n(t, x) := (\Psi_n(t))^{1/p} e^{i\psi_n(t)x} r_y(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.14)$$

belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (5.11) it follows that

$$\psi_n(t) = \log \frac{\gamma_{n-1}[\gamma(t)]}{t} = \log \frac{\gamma_{n-1}[\gamma(t)]}{\gamma(t)} + \log \frac{\gamma(t)}{t} = \psi_{n-1}[\gamma(t)] + \psi(t).$$

Therefore

$$\psi'_n(t) = \psi'_{n-1}[\gamma(t)]\gamma'(t) + \psi'(t). \quad (5.15)$$

By using $\gamma(t) = te^{\psi(t)}$ and $\gamma'(t) = \Psi(t)e^{\psi(t)}$, from (5.11) and (5.15) we get

$$\begin{aligned} \Psi_n(t) &= t\psi'_{n-1}[\gamma(t)]\Psi(t)e^{\psi(t)} + (1 + t\psi'(t)) \\ &= \Psi(t)(1 + \gamma(t)\psi'_{n-1}[\gamma(t)]) = \Psi(t)\Psi_{n-1}[\gamma(t)]. \end{aligned}$$

From this identity by induction we get

$$\Psi_n(t) = \prod_{k=0}^{n-1} \Psi[\gamma_k(t)], \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}. \quad (5.16)$$

From (5.12)–(5.14) and (5.16) we get

$$(vU_\gamma)^n R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{c}_n) \Phi, \quad n \in \mathbb{N}, \quad (5.17)$$

where the function \mathfrak{c}_n is given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}_n(t, x) := a_n(t)\mathfrak{b}_n(t, x) \quad (5.18)$$

with

$$a_n(t) := \prod_{k=0}^{n-1} v[\gamma_k(t)] (\Psi[\gamma_k(t)])^{1/p}, \quad \mathfrak{b}_n(t, x) := e^{i\psi_n(t)x} r_y(x). \quad (5.19)$$

By the hypothesis, $v \in SO(\mathbb{R}_+)$. On the other hand, $\Psi \in SO(\mathbb{R}_+)$ in view of Lemma 2.3. Hence $\Psi^{1/p} \in SO(\mathbb{R}_+)$. Then, due to Lemmas 2.4 and 2.5, $a_n \in SO(\mathbb{R}_+)$ for all $n \in \mathbb{N}$. Therefore, from Lemma 5.1 we obtain that $\mathfrak{a}_n(t, x) := a_n(t), (t, x) \in \mathbb{R}_+ \times \mathbb{R}$, belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. On the other hand, by Lemma 5.2, $\mathfrak{b}_n \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Thus, $\mathfrak{c}_n = a_n \mathfrak{b}_n$ belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ for every $n \in \mathbb{N}$.

Following the proof of [6, Lemma 2.1] (see also [1, Theorem 2.2]), let us show that

$$\limsup_{n \rightarrow \infty} \|a_n\|_{C_b(\mathbb{R}_+)}^{1/n} < 1. \quad (5.20)$$

By Lemmas 2.2 and 2.3,

$$\lim_{t \rightarrow s} \Psi(t) = 1 + \lim_{t \rightarrow s} t\psi'(t) = 1, \quad s \in \{0, \infty\}. \quad (5.21)$$

If $1 \gg v$, then

$$\limsup_{t \rightarrow s} |v(t)| < 1, \quad s \in \{0, \infty\}. \quad (5.22)$$

From (5.21)–(5.22) it follows that

$$L^*(s) := \limsup_{t \rightarrow s} |v(t)(\Psi(t))^{1/p}| < 1, \quad s \in \{0, \infty\}.$$

Fix $\varepsilon > 0$ such that $L^*(s) + \varepsilon < 1$ for $s \in \{0, \infty\}$. By the definition of $L^*(s)$, there exist points $t_1, t_2 \in \mathbb{R}_+$ such that

$$\begin{aligned} |v(t)(\Psi(t))^{1/p}| &< L^*(0) + \varepsilon & \text{for } t \in (0, t_1), \\ |v(t)(\Psi(t))^{1/p}| &< L^*(\infty) + \varepsilon & \text{for } t \in (t_2, \infty). \end{aligned} \quad (5.23)$$

The mapping γ has no fixed points other than 0 and ∞ . Hence, either $\gamma(t) > t$ or $\gamma(t) < t$ for all $t \in \mathbb{R}_+$. For definiteness, suppose that $\gamma(t) > t$ for all $t \in \mathbb{R}_+$. Then there exists a number $k_0 \in \mathbb{N}$ such that $\gamma_{k_0}(t_1) \in (t_2, \infty)$. Put

$$M_1 := \sup_{t \in \mathbb{R}_+} |v(t)(\Psi(t))^{1/p}|, \quad M_2 := \sup_{t \in \mathbb{R}_+ \setminus [t_1, \gamma_{k_0}(t_1)]} |v(t)(\Psi(t))^{1/p}|.$$

Since $v\Psi^{1/p} \in SO(\mathbb{R}_+)$, we have $M_1 < \infty$. Moreover, from (5.23) we obtain

$$M_2 \leq \max(L^*(0), L^*(\infty)) + \varepsilon < 1.$$

Then, for every $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\begin{aligned} |a_n(t)| &= \prod_{k=0}^{n-1} |v[\gamma_k(t)](\Psi[\gamma_k(t)])^{1/p}| \\ &\leq M_1^{k_0} M_2^{n-k_0} \leq M_1^{k_0} (\max(L^*(0), L^*(\infty)) + \varepsilon)^{n-k_0}. \end{aligned}$$

From here we immediately get (5.20).

Now let us show that

$$\limsup_{n \rightarrow \infty} \|\mathbf{b}_n\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}^{1/n} \leq 1. \quad (5.24)$$

By Lemma 5.2, there exists a constant $C(y) \in (0, \infty)$ depending only on y such that for all $n \in \mathbb{N}$,

$$\|\mathbf{b}_n\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \leq C(y) \left(1 + \sup_{t \in \mathbb{R}_+} |\psi_n(t)| \right). \quad (5.25)$$

From (5.11) we obtain

$$\psi_n(t) = \log \left(\prod_{k=0}^{n-1} \frac{\gamma[\gamma_k(t)]}{\gamma_k(t)} \right) = \sum_{k=0}^{n-1} \log \frac{\gamma[\gamma_k(t)]}{\gamma_k(t)} = \sum_{k=0}^{n-1} \psi[\gamma_k(t)]. \quad (5.26)$$

Let

$$M_3 := \sup_{t \in \mathbb{R}_+} |\psi(t)|.$$

Since γ_k is a diffeomorphism of \mathbb{R}_+ onto itself for every $k \in \mathbb{Z}$, we have

$$M_3 = \sup_{t \in \mathbb{R}_+} |\psi(t)| = \sup_{t \in \mathbb{R}_+} |\psi[\gamma(t)]| = \dots = \sup_{t \in \mathbb{R}_+} |\psi[\gamma_{n-1}(t)]|. \quad (5.27)$$

From (5.25)–(5.27) we obtain

$$\|\mathbf{b}_n\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \leq C(y)(1 + M_3 n), \quad n \in \mathbb{N},$$

which implies (5.24). Combining (5.18), (5.20), and (5.24), we arrive at

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\mathbf{c}_n\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}^{1/n} &\leq \left(\limsup_{n \rightarrow \infty} \|a_n\|_{C_b(\mathbb{R}_+)}^{1/n} \right) \\ &\times \left(\limsup_{n \rightarrow \infty} \|\mathbf{b}_n\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}^{1/n} \right) < 1. \end{aligned}$$

This shows that the series $\sum_{n=0}^{\infty} \mathbf{c}_n$ is absolutely convergent in the norm of $C_b(\mathbb{R}_+, V(\mathbb{R}))$. From (5.18)–(5.19) and (5.26) we get for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $n \in \mathbb{N}$,

$$\mathbf{c}_n(t, x) = \left(\prod_{k=0}^{n-1} v[\gamma_k(t)] (\Psi[\gamma_k(t)])^{1/p} e^{i\psi[\gamma_k(t)]x} \right) r_y(x). \quad (5.28)$$

We have already shown that \mathbf{c}_0 given by (5.10) and \mathbf{c}_n , $n \in \mathbb{N}$, given by (5.28) belong to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Thus $\mathbf{c} := \sum_{n=0}^{\infty} \mathbf{c}_n$ is given by (5.7) and it belongs to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

From (5.9), (5.17) and Theorem 4.1 we get

$$\begin{aligned} \left\| \Phi^{-1} \operatorname{Op}(\mathbf{c}) \Phi - \sum_{n=0}^N (vU_{\gamma})^n R_y \right\|_{\mathcal{B}(L^p(\mathbb{R}_+))} &= \left\| \Phi^{-1} \left(\mathbf{c} - \sum_{n=0}^N \mathbf{c}_n \right) \Phi \right\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \\ &\leq C_p \left\| \mathbf{c} - \sum_{n=0}^N \mathbf{c}_n \right\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \\ &= o(1) \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} (vU_{\gamma})^n R_y = \Phi^{-1} \operatorname{Op}(\mathbf{c}) \Phi.$$

Combining this identity with (5.8), we arrive at (5.6). Part (b) is proved.

(c) From (5.10) and (5.28) it follows that $\mathbf{c}_n(t, \pm\infty) = 0$ for $n \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}_+$. Then, in view of Lemma 4.4,

$$\mathbf{c}(t, \pm\infty) = 0, \quad t \in \mathbb{R}_+. \quad (5.29)$$

Since $v, \psi \in SO(\mathbb{R}_+)$, from Lemma 5.1 it follows that the functions

$$\mathbf{v}(t, x) = v(t), \quad \tilde{\psi}(t, x) := \psi(t), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.30)$$

belong to the Banach algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Consider the countable family $\{\mathbf{v}, \tilde{\psi}, \mathbf{c}\} \cup \{\mathbf{c}_n\}_{n=0}^{\infty}$ of functions in $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Fix $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. By Lemma 4.2 and (5.30), there is a sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ and functions $\mathbf{c}(\xi, \cdot) \in V(\mathbb{R}_+)$, $\mathbf{c}_n(\xi, \cdot) \in V(\mathbb{R}_+)$, $n \in \mathbb{N} \cup \{0\}$, such that

$$\lim_{j \rightarrow \infty} t_j = s, \quad v(\xi) = \lim_{j \rightarrow \infty} v(t_j), \quad \psi(\xi) = \lim_{j \rightarrow \infty} \psi(t_j), \quad (5.31)$$

and for $n \in \mathbb{N} \cup \{0\}$ and $x \in \overline{\mathbb{R}}$,

$$\mathbf{c}_n(\xi, x) = \lim_{j \rightarrow \infty} \mathbf{c}_n(t_j, x), \quad \mathbf{c}(\xi, x) = \lim_{j \rightarrow \infty} \mathbf{c}(t_j, x). \quad (5.32)$$

From (5.29) and the second limit in (5.32) we get

$$\mathfrak{c}(\xi, \pm\infty) = \lim_{j \rightarrow \infty} \mathfrak{c}(t_j, \pm\infty) = 0. \quad (5.33)$$

Trivially,

$$\mathfrak{c}_0(\xi, x) = r_y(x), \quad (\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \overline{\mathbb{R}}. \quad (5.34)$$

From Lemmas 2.2 and 2.3 we obtain

$$\lim_{t \rightarrow s} (\Psi(t))^{1/p} = 1, \quad s \in \{0, \infty\}. \quad (5.35)$$

From Lemma 2.5 it follows that for $k \in \mathbb{N}$,

$$\lim_{j \rightarrow \infty} v(t_j) = \lim_{j \rightarrow \infty} v[\gamma_k(t_j)], \quad \lim_{j \rightarrow \infty} \psi(t_j) = \lim_{j \rightarrow \infty} \psi[\gamma_k(t_j)]. \quad (5.36)$$

Combining (5.28), (5.31), the first limit in (5.32), and (5.35)–(5.36), we get for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{c}_n(\xi, x) &= \lim_{j \rightarrow \infty} \left(\prod_{k=0}^{n-1} v[\gamma_k(t_j)] (\Psi[\gamma_k(t_j)])^{1/p} e^{i\psi[\gamma_k(t_j)]x} \right) r_y(x) \\ &= (v(\xi) e^{i\psi(\xi)x})^n r_y(x). \end{aligned} \quad (5.37)$$

From (5.34), (5.37), and Lemma 4.4 we obtain

$$\mathfrak{c}(\xi, x) = \sum_{n=0}^{\infty} (v(\xi) e^{i\psi(\xi)x})^n r_y(x). \quad (5.38)$$

Since $1 \gg v$, we have

$$\limsup_{t \rightarrow s} |v(t)| < 1, \quad s \in \{0, \infty\},$$

whence, in view of Lemma 2.1, we obtain

$$|v(\xi) e^{i\psi(\xi)x}| \leq \max_{s \in \{0, \infty\}} \left(\limsup_{t \rightarrow s} |v(t)| \right) < 1.$$

Therefore,

$$\sum_{n=0}^{\infty} (v(\xi) e^{i\psi(\xi)x})^n = (1 - v(\xi) e^{i\psi(\xi)x})^{-1}. \quad (5.39)$$

From (5.38) and (5.39) we get

$$\mathfrak{c}(\xi, x) = (1 - v(\xi) e^{i\psi(\xi)x})^{-1} r_y(x), \quad (\xi, x) \in \Delta \times \mathbb{R}. \quad (5.40)$$

Combining (5.29), (5.33), and (5.40), we arrive at the assertion of part (c). \square

6. Fredholmness and Index of the Operator V

6.1. First Step of Regularization

Lemma 6.1. *Let $\alpha, \beta \in SOS(\mathbb{R}_+)$ and let $c, d \in SO(\mathbb{R}_+)$ be such that $1 \gg c$ and $1 \gg d$. Then for every $\mu \in [0, 1]$ and $y \in (1, \infty)$ the following statements hold:*

- (a) *the operators $I - \mu c U_\alpha$ and $I - \mu d U_\beta$ are invertible on $L^p(\mathbb{R}_+)$;*
- (b) *for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, the functions*

$$\mathfrak{a}_{\mu,y}^{c,\alpha}(t, x) := (1 - \mu c(t)(\Omega(t))^{1/p} e^{i\omega(t)x}) r_y(x), \quad (6.1)$$

$$\mathfrak{a}_{\mu,y}^{d,\beta}(t, x) := (1 - \mu d(t)(H(t))^{1/p} e^{i\eta(t)x}) r_y(x) \quad (6.2)$$

and

$$\begin{aligned} \mathfrak{c}_{\mu,y}^{c,\alpha}(t, x) := & r_y(x) \\ & + \sum_{n=1}^{\infty} \mu^n \left(\prod_{k=0}^{n-1} c[\alpha_k(t)] (\Omega[\alpha_k(t)])^{1/p} e^{i\omega[\alpha_k(t)]x} \right) r_y(x), \end{aligned} \quad (6.3)$$

$$\begin{aligned} \mathfrak{c}_{\mu,y}^{d,\beta}(t, x) := & r_y(x) \\ & + \sum_{n=1}^{\infty} \mu^n \left(\prod_{k=0}^{n-1} d[\beta_k(t)] (H[\beta_k(t)])^{1/p} e^{i\eta[\beta_k(t)]x} \right) r_y(x), \end{aligned} \quad (6.4)$$

with

$$\omega(t) = \log[\alpha(t)/t], \quad \Omega(t) = 1 + t\omega'(t), \quad (6.5)$$

$$\eta(t) = \log[\beta(t)/t], \quad H(t) = 1 + t\eta'(t), \quad (6.6)$$

belong to the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$;

- (c) the operators

$$V_{\mu,y} := (I - \mu c U_\alpha) P_y^+ + (I - \mu d U_\beta) P_y^-, \quad (6.7)$$

$$L_{\mu,y} := (I - \mu c U_\alpha)^{-1} P_y^+ + (I - \mu d U_\beta)^{-1} P_y^- \quad (6.8)$$

are related by

$$V_{\mu,y} L_{\mu,y} \simeq L_{\mu,y} V_{\mu,y} \simeq H_{\mu,y}, \quad (6.9)$$

where

$$H_{\mu,y} := \Phi^{-1} \operatorname{Op}(\mathfrak{h}_{\mu,y}) \Phi \quad (6.10)$$

and the function $\mathfrak{h}_{\mu,y}$, given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\begin{aligned} \mathfrak{h}_{\mu,y}(t, x) := & 1 + \frac{1}{4} [2(r_y(x))^2 \\ & - \mathfrak{a}_{\mu,y}^{d,\beta}(t, x) \mathfrak{c}_{\mu,y}^{c,\alpha}(t, x) - \mathfrak{a}_{\mu,y}^{c,\alpha}(t, x) \mathfrak{c}_{\mu,y}^{d,\beta}(t, x)], \end{aligned} \quad (6.11)$$

belongs to the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. (a) From Lemma 2.7 it follows that the operators

$$I - \mu c U_\alpha, I - \mu d U_\beta \in \mathcal{FO}_{\alpha, \beta} \quad (6.12)$$

are invertible and

$$(I - \mu c U_\alpha)^{-1}, (I - \mu d U_\beta)^{-1} \in \mathcal{FO}_{\alpha, \beta}. \quad (6.13)$$

This completes the proof of part (a).

(b) From Lemma 3.1 it follows that

$$R_y^2 = \Phi^{-1} \text{Co}(r_y^2) \Phi = \Phi^{-1} \text{Op}(\mathfrak{r}_y^2) \Phi, \quad (6.14)$$

where $\mathfrak{r}_y(t, x) = r_y(x)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. From Lemma 5.1 we deduce that $\mathfrak{r}_y^2 \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. By Lemma 5.4(a),

$$(I - \mu c U_\alpha) R_y = \Phi^{-1} \text{Op}(\mathfrak{a}_{\mu, y}^{c, \alpha}) \Phi, \quad (6.15)$$

$$(I - \mu d U_\beta) R_y = \Phi^{-1} \text{Op}(\mathfrak{a}_{\mu, y}^{d, \beta}) \Phi, \quad (6.16)$$

where the functions $\mathfrak{a}_{\mu, y}^{c, \alpha}$ and $\mathfrak{a}_{\mu, y}^{d, \beta}$, given by (6.1) and (6.2), respectively, belong to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Lemma 5.5(b) implies that

$$(I - \mu c U_\alpha)^{-1} R_y = \Phi^{-1} \text{Op}(\mathfrak{c}_{\mu, y}^{c, \alpha}) \Phi, \quad (6.17)$$

$$(I - \mu d U_\beta)^{-1} R_y = \Phi^{-1} \text{Op}(\mathfrak{c}_{\mu, y}^{d, \beta}) \Phi, \quad (6.18)$$

where the functions $\mathfrak{c}_{\mu, y}^{c, \alpha}$ and $\mathfrak{c}_{\mu, y}^{d, \beta}$ given by (6.3) and (6.4), respectively, belong to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. In particular, this completes the proof of part (b).

(c) From (6.12)–(6.13) and Lemmas 2.8 and 3.1 it follows that

$$(I - \mu c U_\alpha)^t T \simeq T(I - \mu c U_\alpha)^t, \quad (6.19)$$

$$(I - \mu d U_\beta)^t T \simeq T(I - \mu d U_\beta)^t \quad (6.20)$$

for every $t \in \{-1, 1\}$ and $T \in \{P_y^+, P_y^-, R_y\}$. Applying consecutively relations (6.19)–(6.20) with $T \in \{P_y^+, P_y^-\}$, Lemma 3.2(b), and relations (6.19)–(6.20) with $T = R_y$, we get

$$\begin{aligned} V_{\mu, y} L_{\mu, y} &\simeq (P_y^+)^2 + (I - \mu d U_\beta)(I - \mu c U_\alpha)^{-1} P_y^- P_y^+ \\ &\quad + (P_y^-)^2 + (I - \mu c U_\alpha)(I - \mu d U_\beta)^{-1} P_y^+ P_y^- \\ &= \left(P_y^+ + \frac{R_y^2}{4} \right) - (I - \mu d U_\beta)(I - \mu c U_\alpha)^{-1} \frac{R_y^2}{4} \\ &\quad + \left(P_y^- + \frac{R_y^2}{4} \right) - (I - \mu c U_\alpha)(I - \mu d U_\beta)^{-1} \frac{R_y^2}{4} \\ &\simeq I + \frac{R_y^2}{2} - \frac{1}{4} (I - \mu d U_\beta) R_y (I - \mu c U_\alpha)^{-1} R_y \\ &\quad - \frac{1}{4} (I - \mu c U_\alpha) R_y (I - \mu d U_\beta)^{-1} R_y. \end{aligned} \quad (6.21)$$

Applying equalities (6.15)–(6.18) to (6.21), we obtain

$$\begin{aligned} V_{\mu, y} L_{\mu, y} &\simeq I + \frac{1}{2} \Phi^{-1} \text{Op}(\mathfrak{r}_y^2) \Phi - \frac{1}{4} \Phi^{-1} \text{Op}(\mathfrak{a}_{\mu, y}^{d, \beta}) \text{Op}(\mathfrak{c}_{\mu, y}^{c, \alpha}) \Phi \\ &\quad - \frac{1}{4} \Phi^{-1} \text{Op}(\mathfrak{a}_{\mu, y}^{c, \alpha}) \text{Op}(\mathfrak{c}_{\mu, y}^{d, \beta}) \Phi. \end{aligned} \quad (6.22)$$

From Theorem 4.5 we get

$$\text{Op}(\mathfrak{a}_{\mu,y}^{d,\beta}) \text{Op}(\mathfrak{c}_{\mu,y}^{c,\alpha}) \simeq \text{Op}(\mathfrak{a}_{\mu,y}^{d,\beta} \mathfrak{c}_{\mu,y}^{c,\alpha}), \quad (6.23)$$

$$\text{Op}(\mathfrak{a}_{\mu,y}^{c,\alpha}) \text{Op}(\mathfrak{c}_{\mu,y}^{d,\beta}) \simeq \text{Op}(\mathfrak{a}_{\mu,y}^{c,\alpha} \mathfrak{c}_{\mu,y}^{d,\beta}). \quad (6.24)$$

Combining (6.22)–(6.24), we arrive at

$$V_{\mu,y} L_{\mu,y} \simeq \Phi^{-1} \text{Op}(\mathfrak{h}_{\mu,y}) \Phi,$$

where the function $\mathfrak{h}_{\mu,y}$, given by (6.11), belongs to the algebra $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ because the functions (6.1)–(6.4) lie in this algebra in view of part (b). Analogously, it can be shown that

$$L_{\mu,y} V_{\mu,y} \simeq \Phi^{-1} \text{Op}(\mathfrak{h}_{\mu,y}) \Phi,$$

which completes the proof. \square

6.2. Fredholmness of the Operator $H_{\mu,2}$

In this subsection we will prove that the operators $H_{\mu,2}$ given by (6.10) are Fredholm for every $\mu \in [0, 1]$. To this end, we will use Theorem 4.7.

First we represent boundary values of $\mathfrak{h}_{\mu,y}$ in a way, which is convenient for further analysis.

Lemma 6.2. *Let $\alpha, \beta \in SOS(\mathbb{R}_+)$ and let $c, d \in SO(\mathbb{R}_+)$ be such that $1 \gg c$ and $1 \gg d$. If $\mathfrak{h}_{\mu,y}$ is given by (6.11) and (6.1)–(6.6), then for every $\mu \in [0, 1]$ and $y \in (1, \infty)$, we have*

$$\mathfrak{h}_{\mu,y}(\xi, x) = \begin{cases} v_{\mu,y}(\xi, x) \ell_{\mu,y}(\xi, x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 1, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}, \end{cases}$$

where

$$v_{\mu,y}(\xi, x) := (1 - \mu c(\xi) e^{i\omega(\xi)x}) p_y^+(x) + (1 - \mu d(\xi) e^{i\eta(\xi)x}) p_y^-(x), \quad (6.25)$$

$$\ell_{\mu,y}(\xi, x) := (1 - \mu c(\xi) e^{i\omega(\xi)x})^{-1} p_y^+(x) + (1 - \mu d(\xi) e^{i\eta(\xi)x})^{-1} p_y^-(x) \quad (6.26)$$

for $(\xi, x) \in \Delta \times \mathbb{R}$.

Proof. From (6.11), Lemmas 4.3, 5.4(b), and 5.5(c) it follows that

$$\mathfrak{h}_{\mu,y}(\xi, x) = 1 \quad \text{for } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}$$

and

$$\begin{aligned} \mathfrak{h}_{\mu,y}(\xi, x) = & 1 + \frac{1}{4} [2(r_y(x))^2 \\ & - (1 - \mu d(\xi) e^{i\eta(\xi)x}) r_y(x) (1 - \mu c(\xi) e^{i\omega(\xi)x})^{-1} r_y(x) \\ & - (1 - \mu c(\xi) e^{i\omega(\xi)x}) r_y(x) (1 - \mu d(\xi) e^{i\eta(\xi)x})^{-1} r_y(x)] \end{aligned}$$

for $(\xi, x) \in \Delta \times \mathbb{R}$. By Lemma 3.2(a),

$$\begin{aligned}
\mathfrak{h}_{\mu,y}(\xi, x) &= \\
&= \left(p_y^+(x) + \frac{(r_y(x))^2}{4} \right) - (1 - \mu d(\xi) e^{i\eta(\xi)x})(1 - \mu c(\xi) e^{i\omega(\xi)x})^{-1} \frac{(r_y(x))^2}{4} \\
&\quad + \left(p_y^-(x) + \frac{(r_y(x))^2}{4} \right) - (1 - \mu c(\xi) e^{i\omega(\xi)x})(1 - \mu d(\xi) e^{i\eta(\xi)x})^{-1} \frac{(r_y(x))^2}{4} \\
&= (p_y^+(x))^2 + (1 - \mu d(\xi) e^{i\eta(\xi)x})(1 - \mu c(\xi) e^{i\omega(\xi)x})^{-1} p_y^-(x) p_y^+(x) \\
&\quad + (p_y^-(x))^2 + (1 - \mu c(\xi) e^{i\omega(\xi)x})(1 - \mu d(\xi) e^{i\eta(\xi)x})^{-1} p_y^+(x) p_y^-(x) \\
&= v_{\mu,y}(\xi, x) \ell_{\mu,y}(\xi, x)
\end{aligned}$$

for $(\xi, x) \in \Delta \times \mathbb{R}$, which completes the proof. \square

We were unable to prove that $\mathfrak{h}_{\mu,y}$ satisfies the hypotheses of Theorem 4.7 for every $y \in (1, \infty)$ or at least for $y = p$. However, the very special form of the ranges of $v_{\mu,2}$ and $\ell_{\mu,2}$ given by (6.25) and (6.26), respectively, allows us to prove that $v_{\mu,2}$ and $\ell_{\mu,2}$ are separated from zero for all $\mu \in [0, 1]$, and thus $\mathfrak{h}_{\mu,2}$ satisfies the assumptions of Theorem 4.7.

Lemma 6.3. *Let $\alpha, \beta \in SOS(\mathbb{R}_+)$ and let $c, d \in SO(\mathbb{R}_+)$ be such that $1 \gg c$ and $1 \gg d$. Then for every $\mu \in [0, 1]$ the operator $H_{\mu,2}$ given by (6.10) is Fredholm on $L^p(\mathbb{R}_+)$.*

Proof. By Lemma 6.2, for the function $\mathfrak{h}_{\mu,2}$ defined by (6.11) and (6.1)–(6.6) we have

$$\mathfrak{h}_{\mu,2}(\xi, x) = 1 \neq 0 \quad \text{for } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\} \quad (6.27)$$

and

$$\mathfrak{h}_{\mu,2}(\xi, x) = v_{\mu,2}(\xi, x) \ell_{\mu,2}(\xi, x) \quad \text{for } (\xi, x) \in \Delta \times \mathbb{R},$$

where $v_{\mu,2}$ and $\ell_{\mu,2}$ are defined by (6.25) and (6.26), respectively. From Lemmas 2.9 and 2.10 it follows that for each $\xi \in \Delta$ the ranges of the continuous functions $v_{\mu,2}(\xi, \cdot)$ and $\ell_{\mu,2}(\xi, \cdot)$ defined on \mathbb{R} lie in the half-plane

$$\mathcal{H}^{\mu,\xi} := \{z \in \mathbb{C} : \operatorname{Re} z > 1 - \mu \max(|c(\xi)|, |d(\xi)|)\}.$$

From Lemma 2.1 we get

$$\begin{aligned}
C(\Delta) &:= \sup_{\xi \in \Delta} |c(\xi)| = \max_{s \in \{0, \infty\}} \left(\limsup_{t \rightarrow s} |c(t)| \right), \\
D(\Delta) &:= \sup_{\xi \in \Delta} |d(\xi)| = \max_{s \in \{0, \infty\}} \left(\limsup_{t \rightarrow s} |d(t)| \right).
\end{aligned}$$

Since $1 \gg c$ and $1 \gg d$, we see that $C(\Delta) < 1$ and $D(\Delta) < 1$. Therefore, for every $\xi \in \Delta$ and $\mu \in [0, 1]$, the half-plane $\mathcal{H}^{\mu,\xi}$ is contained in the half-plane

$$\{z \in \mathbb{C} : \operatorname{Re} z > 1 - \max(|C(\Delta)|, |D(\Delta)|)\}$$

and the origin does not lie in the latter half-plane. Thus

$$\mathfrak{h}_{\mu,2}(\xi, x) = v_{\mu,2}(\xi, x) \ell_{\mu,2}(\xi, x) \neq 0 \quad \text{for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (6.28)$$

From (6.27)–(6.28) and Theorem 4.7 we obtain that the operator $H_{\mu,2}$ is Fredholm on $L^p(\mathbb{R}_+)$. \square

6.3. Proof of the Main Result

For $\mu \in [0, 1]$, consider the operators $V_{\mu,2}$ and $L_{\mu,2}$ defined by (6.7) and (6.8), respectively. It is obvious that $V_{0,2} = P_y^+ + P_y^- = I$ and $V_{1,2} = V$. By Lemma 6.1(c),

$$V_{\mu,2}L_{\mu,2} \simeq L_{\mu,2}V_{\mu,2} \simeq H_{\mu,2}, \quad \mu \in [0, 1], \quad (6.29)$$

where the operator $H_{\mu,2}$ given by (6.10) is Fredholm in view of Lemma 6.3.

Let $H_{\mu,2}^{(-1)}$ be a regularizer for $H_{\mu,2}$. From (6.29) it follows that

$$V_{\mu,2}(L_{\mu,2}H_{\mu,2}^{(-1)}) \simeq I, \quad (H_{\mu,2}^{(-1)}L_{\mu,2})V_{\mu,2} \simeq I, \quad \mu \in [0, 1]. \quad (6.30)$$

Thus, $L_{\mu,2}H_{\mu,2}^{(-1)}$ is a right regularizer for $V_{\mu,2}$ and $H_{\mu,2}^{(-1)}L_{\mu,2}$ is a left regularizer for $V_{\mu,2}$. Therefore, $V_{\mu,2}$ is Fredholm for every $\mu \in [0, 1]$. It is obvious that the operator-valued function $\mu \mapsto V_{\mu,2} \in \mathcal{B}(L^p(\mathbb{R}_+))$ is continuous on $[0, 1]$. Hence the operators $V_{\mu,2}$ belong to the same connected component of the set of all Fredholm operators. Therefore all $V_{\mu,2}$ have the same index (see, e.g., [4, Section 4.10]). Since $V_{0,2} = I$, we conclude that

$$\text{Ind } V = \text{Ind } V_{1,2} = \text{Ind } V_{0,2} = \text{Ind } I = 0,$$

which completes the proof of Theorem 1.1. \square

7. Regularization of the Operator W

7.1. Regularizers of the Operator W

As a by-product of the proof of Section 6, we can describe all regularizers of a slightly more general operator W .

Theorem 7.1. *Let $1 < p < \infty$, $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, and $\alpha, \beta \in \text{SOS}(\mathbb{R}_+)$. Suppose $c, d \in \text{SO}(\mathbb{R}_+)$ are such that $1 \gg c$ and $1 \gg d$. Then the operator W given by*

$$W := (I - cU_\alpha^{\varepsilon_1})P_2^+ + (I - dU_\beta^{\varepsilon_2})P_2^-$$

is Fredholm on the space $L^p(\mathbb{R}_+)$ and $\text{Ind } W = 0$. Moreover, each regularizer $W^{(-1)}$ of the operator W is of the form

$$W^{(-1)} = [\Phi^{-1} \text{Op}(\mathfrak{f})\Phi] \cdot [(I - cU_\alpha^{\varepsilon_1})^{-1}P_2^+ + (I - dU_\beta^{\varepsilon_2})^{-1}P_2^-] + K, \quad (7.1)$$

where $K \in \mathcal{K}(L^p(\mathbb{R}_+))$ and $\mathfrak{f} \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that

$$\mathfrak{f}(\xi, x) = \begin{cases} \frac{1}{w(\xi, x)\ell(\xi, x)}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 1, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}, \end{cases} \quad (7.2)$$

where

$$w(\xi, x) := (1 - c(\xi)e^{i\varepsilon_1\omega(\xi)x})p_2^+(x) + (1 - d(\xi)e^{i\varepsilon_2\eta(\xi)x})p_2^-(x) \neq 0, \quad (7.3)$$

$$\ell(\xi, x) := \frac{p_2^+(x)}{1 - c(\xi)e^{i\varepsilon_1\omega(\xi)x}} + \frac{p_2^-(x)}{1 - d(\xi)e^{i\varepsilon_2\eta(\xi)x}} \neq 0 \quad (7.4)$$

for $(\xi, x) \in \Delta \times \mathbb{R}$ with $\omega(t) := \log[\alpha(t)/t]$ and $\eta(t) := \log[\beta(t)/t]$ for $t \in \mathbb{R}_+$.

Proof. Since $\alpha, \beta \in SOS(\mathbb{R}_+)$, from Lemma 2.5 it follows that α_{-1} and β_{-1} also belong to $SOS(\mathbb{R}_+)$. Taking into account that $U_\alpha^{\varepsilon_1} = U_{\alpha_{\varepsilon_1}}$ and $U_\beta^{\varepsilon_2} = U_{\beta_{\varepsilon_2}}$, from Theorem 1.1 we deduce that the operator W is Fredholm and $\text{Ind } W = 0$. Further, from (6.30) and Lemma 6.3 it follows that each regularizer $W^{(-1)}$ of W is of the form

$$W^{(-1)} = H^{(-1)}L + K_1, \quad (7.5)$$

where $K_1 \in \mathcal{K}(L^p(\mathbb{R}_+))$,

$$L := (I - cU_\alpha^{\varepsilon_1})^{-1}P_2^+ + (I - dU_\beta^{\varepsilon_2})^{-1}P_2^-, \quad (7.6)$$

and $H^{(-1)}$ is a regularizer of the Fredholm operator H given by

$$H := \Phi^{-1} \text{Op}(\mathfrak{h})\Phi,$$

where $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{h}(t, x) := 1 + \frac{1}{4} [2(r_2(x))^2 - \mathfrak{a}_{1,2}^{d,\beta_{\varepsilon_2}}(t, x)\mathfrak{c}_{1,2}^{c,\alpha_{\varepsilon_1}}(t, x) - \mathfrak{a}_{1,2}^{c,\alpha_{\varepsilon_1}}(t, x)\mathfrak{c}_{1,2}^{d,\beta_{\varepsilon_2}}(t, x)],$$

and the functions $\mathfrak{a}_{1,2}^{c,\alpha_{\varepsilon_1}}$, $\mathfrak{c}_{1,2}^{c,\alpha_{\varepsilon_1}}$ and $\mathfrak{a}_{1,2}^{d,\beta_{\varepsilon_2}}$, $\mathfrak{c}_{1,2}^{d,\beta_{\varepsilon_2}}$ are given by (6.1)–(6.2) and (6.3)–(6.4) with α and β replaced by α_{ε_1} and β_{ε_2} , respectively.

Taking into account Lemma 2.6, by analogy with Lemma 6.2 we get

$$\mathfrak{h}(\xi, x) = \begin{cases} w(\xi, x)\ell(\xi, x), & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 1, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}. \end{cases} \quad (7.7)$$

By Theorem 4.7(b), each regularizer $H^{(-1)}$ of the Fredholm operator H is of the form

$$H^{(-1)} = \Phi^{-1} \text{Op}(\mathfrak{f})\Phi + K_2, \quad (7.8)$$

where $K_2 \in \mathcal{K}(L^p(\mathbb{R}_+))$ and $\mathfrak{f} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that

$$\begin{aligned} \mathfrak{f}(t, \pm\infty) &= 1/\mathfrak{h}(t, \pm\infty) & \text{for all } t \in \mathbb{R}_+, \\ \mathfrak{f}(\xi, x) &= 1/\mathfrak{h}(\xi, x) & \text{for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \end{aligned} \quad (7.9)$$

From (7.5)–(7.6) and (7.8) we get (7.1). Combining (7.7) and (7.9), we arrive at (7.2). \square

7.2. One Useful Consequence of Regularization of W

Theorem 7.2. *Under the assumptions of Theorem 7.1, for every $y \in (1, \infty)$ there exists a function $\mathfrak{g}_y \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that*

$$(\Phi^{-1} \operatorname{Op}(\mathfrak{g}_y) \Phi) W \simeq R_y \quad (7.10)$$

and

$$\mathfrak{g}_y(\xi, x) = \begin{cases} \frac{r_y(x)}{w(\xi, x)}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}, \end{cases}$$

where the function w is defined for $(\xi, x) \in \Delta \times \mathbb{R}$ by (7.3).

Proof. From Theorem 7.1 it follows that

$$(\Phi^{-1} \operatorname{Op}(\mathfrak{f}) \Phi) L W R_y \simeq R_y, \quad (7.11)$$

where L is given by (7.6) and $\mathfrak{f} \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ satisfies (7.2). From Lemmas 2.8 and 3.1 we get

$$W R_y \simeq R_y W. \quad (7.12)$$

Lemmas 3.1 and 5.5(a)–(b) imply that

$$\begin{aligned} L R_y &= (I - c U_\alpha^{\varepsilon_1})^{-1} R_y P_2^+ + (I - d U_\beta^{\varepsilon_2})^{-1} R_y P_2^- \\ &= \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}}) \operatorname{Co}(p_2^+) \Phi + \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}}) \operatorname{Co}(p_2^-) \Phi, \end{aligned} \quad (7.13)$$

where the functions $\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}}$ and $\mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}}$, given by (6.3) and (6.4) with α and β replaced by α_{ε_1} and β_{ε_2} , respectively, belong to $\tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (7.13) and Lemmas 5.1 and 4.6 we obtain

$$L R_y = \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}} p_2^+ + \mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}} p_2^-) \Phi, \quad (7.14)$$

where $\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}} p_2^+ + \mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}} p_2^- \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (7.11)–(7.12), (7.14), and Theorem 4.5 we get (7.10) with

$$\mathfrak{g}_y := \mathfrak{f}(\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}} p_2^+ + \mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}} p_2^-) \in \tilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})). \quad (7.15)$$

Obviously,

$$p_2^\pm(\pm\infty) = 1, \quad p_2^\pm(\mp\infty) = 0. \quad (7.16)$$

By Lemmas 2.6 and 5.5(c),

$$\mathfrak{c}_{1,y}^{c,\alpha_{\varepsilon_1}}(\xi, x) = \begin{cases} \frac{r_y(x)}{1 - c(\xi) e^{i \varepsilon_1 \omega(\xi) x}}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}, \end{cases} \quad (7.17)$$

$$\mathfrak{c}_{1,y}^{d,\beta_{\varepsilon_2}}(\xi, x) = \begin{cases} \frac{r_y(x)}{1 - d(\xi) e^{i \varepsilon_2 \eta(\xi) x}}, & \text{if } (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}. \end{cases} \quad (7.18)$$

From (7.15)–(7.18), (7.2)–(7.4), and Lemma 4.3 we get

$$\mathfrak{g}_y(\xi, x) = 0 \quad \text{for } (\xi, x) \in (\mathbb{R}_+ \cup \Delta) \times \{\pm\infty\}$$

and

$$\begin{aligned}\mathfrak{g}_y(\xi, x) &= \mathfrak{f}(\xi, x) \left(\frac{r_y(x)p_2^+(x)}{1 - c(\xi)e^{i\varepsilon_1\omega(\xi)x}} + \frac{r_y(x)p_2^-(x)}{1 - d(\xi)e^{i\varepsilon_2\eta(\xi)x}} \right) \\ &= \frac{\ell(\xi, x)r_y(x)}{w(\xi, x)\ell(\xi, x)} = \frac{r_y(x)}{w(\xi, x)}\end{aligned}$$

for $(\xi, x) \in \Delta \times \mathbb{R}$. \square

Relation (7.10) will play an important role in the proof of an index formula for the operator N in [11].

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